Topological Quantum Computing: Theory

Outline
I. Non-Abelian anyons / TQC overview

II. Toy Models
   A. Kitaev chain
   B. 2D topo SC

III. Experimental Blueprints

IV. Detection

V. Status + Outlook

Possible exchange statistics
\[ \psi \rightarrow \pm \psi \] (bosons/Fermions; all elementary particles)
\[ \psi \rightarrow e^{i\theta} \psi \] (Abelian anyons)

Focus here
\[ \psi_i \rightarrow U_{ij} \psi_j \] (non-Abelian anyons)

(Ising) non-Abelian anyons

Three hallmarks:
1. Ground state degeneracy gap
   \[ \text{Fermions/gd. states} \]
   \[ \text{locally indistinguishable robust} \]
   \[ \Rightarrow \text{anyons carry unusual few-energy degrees of freedom - "Majorana modes"} \]

Fault-tolerant Qubits
[ motivate through absence of depolarizing ]
cF. Free spin $\frac{1}{2}$'s:

\[ \begin{array}{cc}
\uparrow & \uparrow \\
\downarrow & \downarrow \\
\end{array} \Rightarrow 2^{N_{\text{spin}}} \text{ gd. states, but qualitatively different!}
- locally distinguishable, not robust
- e.g., measure magnetization w/ Zeeman field

2. Non-Abelian statistics

- adiabatic braid

\[ \begin{array}{cc}
U_i & \Rightarrow U_{ij} U_j \\
\end{array} \]

- braid matrix - rigid!
- [i.e., depends on topology of braid, not detailed path chosen]
- gd. states
- Fault-tolerant Quantum Gates

3. Nontrivial fusion

- \[ \sigma \times \sigma = I + \gamma_F \]
- fermion
- \[ \Rightarrow \text{nontrivial fusion} + \text{vice versa} \]
- can "fuse" in multiple ways
- ... can annihilate
- Readout

Kitaev - we for "topological quantum computation!"

But how to build hardware?
"Fisher plot"

- Exp. Relevance
- Focus here
- Key step: strong vs weak correlation!
- "La La Land"
- Topo SC's
- QH realization? (Moore-Read, 91)
- "Designer" top. SC's (last few years)
- Majorana modes (Read-Green, kitamura '00-02)
- Non-Abelian anyons (ide 80's)
- Conceptual Novelty

Toy Models I: kitaev chain

Spinless fermions on N-site chain + P. B. C.'s (for now).

\[ H = \sum_x \left[ -\mu c_x^+ c_x - \frac{1}{2} \left( t c_x^+ c_{x+2} + \Delta c_x c_{x+2} + h.c. \right) \right] \]

P. B. C. => go to k-space,

\[ H = \sum_{k \in \text{BZ}} \left[ (-\mu - t \cos k) c_k^+ c_k + \left( \frac{i\Delta}{2} \sin k c_k c_{-k} + h.c. \right) \right] \]
Phase Diagram as $\Delta$ of $\mu$?

Note: Can rationalize this distinction by noting that gapped phase have same symmetry. but are separated by critical pt. We focus on same point.

Want to explore universal properties of phases/transitions. Convenient to take

$\Delta = \mu$ hereafter.

$$
H = \sum_x \left[ -\epsilon_x c_x^+ c_x - \frac{1}{\Delta} \left( c_x^+ + c_x \right) \left( c_{x+1}^+ - c_{x+1}^+ \right) \right]
$$
Gapped phases

Take open B.C.'s now + use Majorana rep.:

\[ C_x = \frac{i}{\alpha} \left( \chi_{B_x} + i \chi_{A_x} \right) \]

Majorana ops. (only pairs have well-defined occupation \#\#)

\[ \chi_a = \chi_a^+ \quad \gamma^a = I \]

\[ \{ \chi_{a_1}, \chi_{a_1}', \chi_{a_2}, \chi_{a_2}' \} = 0 \]

(Majorana up. algebra — rep. \(
\chi_x^2 = (\chi_x^+)^2 = 0, \quad \{ \chi_x, \chi_x^+ \} = \varepsilon_{xx'} \))

Note: this is always a legitimate rep. of any ordinary fermion op. like \(c_x\). Does not however guarantee that a system support Majorana-like excitations as we'll see!

Rewrite \(H\):

\[ C_x^+ C_x = \frac{1}{4} \left( \chi_{B_x} - i \chi_{A_x} \right) \left( \chi_{B_x} + i \chi_{A_x} \right) \]

\[ = \frac{1}{4} \left( 1 + i \chi_{B_x} \chi_{A_x} - i \chi_{A_x} \chi_{B_x} \right) \]

\[ = \frac{i}{\alpha} \left( 1 + i \chi_{B_x} \chi_{A_x} \right) \]

\[ (C_x^+ + C_x)(C_x + C_x^+) = i \chi_{B_x} \chi_{A_x+1} \]

\[ H = -\frac{i}{\alpha} \sum_x \left( \mu \chi_{B_x} \chi_{A_x} + t \chi_{B_x} \chi_{A_{x+1}} \right) \]

\[ C_x \text{ fermion} \quad \text{Majorana chain w/ competing dimerization \(\mu, t\)!} \]
For revealing snapshots of gapped phases, examine 2 limits:

(i) \( m < 0, t = 0 \)

Unique g.s. state.

Trivial product state w/ no entanglement between sides. (Vaccum of c^\dagger_0 fermis.)

(ii) \( m = 0, t > 0 \)

Several events a. o.

- \( \gamma_1 = c_1 - c_1^\dagger \)
- \( \gamma_2 = c_0 + c_0^\dagger \)
- Non-local fermion \( d = \frac{\gamma_1 + i\gamma_2}{\alpha} \) can be filled, empty w/ no energy cost.

\( \Rightarrow \) 2-fold topological g.s. state deg.!

\[ 10^\uparrow, d|10\rangle \approx 1^\uparrow \]

[Very unusual - most SC's prefer even parity so that all \( c_0 \)'s can pair. Excitation energy for unpaired \( c_0 \)'s vanishes for topo. reason here.]

- For \( m > 0, \) zero modes decay exponentially into bulk.
- Deg. stable to local perturbations; no local measurement can distinguish gap states. [contrast to deg. \( \uparrow \) \& \( \downarrow \) spin states]

- \( \chi_{1/2} \) are not particles (or quasiparticles)!

- Ends of topo SC \( \cong \) "Ising non-Abelian anyons". Zero-modes are internal deg. of freedom that encode gap state deg.

- Each anyon pair gives 2 gap states.

- Can braid in networks: \( \Psi \rightarrow U_1 \Psi \).

- Braids effectively exchange "half" of one fermion w/ "half" of another—hence non-trivially.

- Fusion channels:

Toyo Phase Transition

- Low-energy physics @ criticality?

\[ H \rightarrow H_{\text{crit}} = -\frac{it}{\alpha} \sum_x \left( \chi_{x, x+1} \chi_{x+1, x} + \chi_{x, x+1} \chi_{x+1, x} \right) = -\frac{i}{\alpha} \sum_x \chi_{x, x+1} (\chi_{x, x+1} - \chi_{x+1, x}) \]

- Continuum limit.
Write $\gamma_{\text{AB}} = \gamma_R \pm \gamma_L \Rightarrow H_{\text{crit}} = \frac{-i\hbar}{\alpha} \int_x (\gamma_R \pm \gamma_L) \partial_x (\gamma_R \pm \gamma_L)$

$\frac{-i\hbar}{\alpha} \int_x (\gamma_R \partial_x \gamma_R - \gamma_L \partial_x \gamma_L + \gamma_R \partial_x \gamma_L - \gamma_L \partial_x \gamma_R)$

So we get $H_{\text{crit}} = \int_x (-i\hbar \gamma_R \partial_x \gamma_R + i\hbar \gamma_L \partial_x \gamma_L)$ (node)

chiral, obey Majorana fermions!

Go to $k$-space: $\gamma_{\text{R/L}}(k) = \int e^{ikx} \gamma_{\text{R/L}}(x)$

[Implicitly thinking about P.B.C.'s again.]

Hermiticity $\Rightarrow \gamma_{\text{R/L}}(k) = \gamma_{\text{R/L}}^*(-k)$ (4b)

$\Rightarrow H_{\text{crit}} = \int_k \left[ \nu k \gamma_R^+(k) \gamma_R(k) - \nu k \gamma_L^+(k) \gamma_L(k) \right]$ (4c)

By (4c), E=0, E=0 states not distinct!

"Half" of usual single-channel wire.

[can be made very precise; e.g., thermal transport exactly half that in a usual wire ($c=\frac{1}{2}$ vs. 1).]

So Majorana physics in kinked chain appears in 2 ways:

(i) Localized zero modes in top phase

(ii) Gapless propagating deg. of freedom at criticality.
Top Models II: 2D Topo. SC's

Build from array of critical Kitaev chains

\[
\begin{align*}
& \psi_x \\
& \psi_y \\
& \psi_z \\
& \psi_k \\
\end{align*}
\]

bulk gapped

unpaired chiral Majorana edge states!

[hallmark of 2D topo SC.]

Note similarly to nontrivial dimension in Kitaev chain.

trivial vacuum

\[
H_{\text{edge}} = \int dx \left(-i v \chi^\dagger \gamma^0 \chi\right)
\]

\[
E = v k
\]

Important Q: Spectrum for finite perimeter \(L\)?

i.e. is \(k\) quantized to (i) \(k_n = \frac{2\pi}{L} n\) (PBC)

or

(ii) \(k_n = \frac{2\pi}{L} (n + \frac{1}{2})\) (ant-\(PBC\))

\[
\begin{align*}
& \text{correct answer;} \\
& \text{PBC ruled out because you can't hope just one Majorana zero mode! (Hilbert space wouldn't make sense.)}
\end{align*}
\]
Drill holes:

\[ \text{thread be flux quantum;} \]

\[ \text{unpaired fermion, get - sign when} \]

\[ \text{encircling flux (Aharonov-Bohm)} \]

Majorana B.C.) shift from anti-periodic \( \rightarrow \) periodic! \[ \Rightarrow \text{zero mode} \ x_1, x_2! \]

Lessons: (i) boundaries of 2D top SC host chiral Majorana fermion, [causes of Majorana end-states in 1D]

(ii) be flux localized, a Majorana zero mode [deeply related to (i)] + : Form Ising non-Abelian anyon

**Experimental Blueprints**

Wanted: "Spineless" 1D, 2D SC's \( \in \) both harbor Ising anyon, albeit in different ways.

**Challenges:** (1) We live in 3D

(2) e^\( -s \)'s carry spin

(3) Nearly all SC's arise from singlet Cooper pairs.

Likely no "intrinsic" realizations in solid state—despite 1000's of known SC's!
Can instead "engineer" topo SC's! O(100) papers revealing various strategies. Most follow a common recipe:

**Step I**

- Use TI boundary
- Break $J$ in low-D systems w/ SOC

![Diagram](image1)

One pair of Fermi pts; one Fermi surface as desired

Solves challenges 1, 2! [in a way that metes challenge 3 "easy"]

**Step II**

Couple systems above to s-wave SC.

![Diagram](image2)

"proximity effect" drives 1D, 2D topo SC!!

Lots of expts followed, but first...
Majorana Detector

Focus on tunneling methods - most common so far.

Primer - SC tunneling as scattering problem

\[ G = \frac{dI}{dV} = 2 \times \frac{e^2}{h} \times (\text{Andreev ref. prob.}) \]

Solve just like potential barrier scattering in AM
- get wave for, extract normal/Andreev ref. coefficients.

Result for topo SC

\[ G = \frac{2e^2}{h} \quad (V \to 0) \]
Experimental Status

Devices built:

(Yaroslav, Konynenberg, Du,...)  (van Herlingen, Hasan,...)

Majorana evidence via tunneling here.

1D wire expts

B < B_c

\begin{align*}
E & = \text{not yet 'splitless'} \\
G & \sim D_{\text{inv}}
\end{align*}

B > B_c

\begin{align*}
E & = \text{topological regime} \\
G & \sim \text{zero-bias peak!}
\end{align*}
Consistent w/ Majorana zero-mode formation, but

- peak height far below $\frac{2e^2}{h}$ in most expts. [except recent unpublished data]
- signatures of phase transition often absent
- other explanations proposed [but harder sell as device quality improves]

My take: analogous features in latest high quality devices likely of Majorana origin. Reasonable alternatives not obvious.

Fe chain expts

Huge virtue: spatial resolution of tunneling conductance.

Zero-bias peaks localized to edges.

Tantalizing Majorana evidence! Future exp/theory will be exciting.

Future Milestones

- Majorana detection
- topological transition
- fusion rules
- prototype top-qubit realization
- non-Abelian statistics
- universal gate set
- fault-tolerant QC

where we are now
intermediate milestones wanted!

where we want to go