6-11-17

Topological Quantum Computing: Theory

Outline

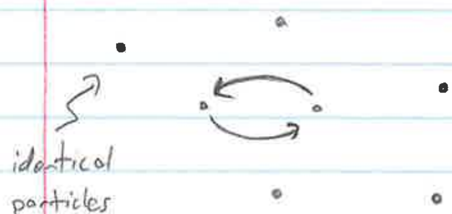
I. Non-Abelian anyons / TQC overview

II. Toy Models
- A. Kitaev chain
- B. 2D topo SC

III. Experimental Blueprints

IV. Detection

V. Status + Outlook



Possible exchange statistics

$\psi \to \pm\psi$ (bosons/fermions; all elementary particles)

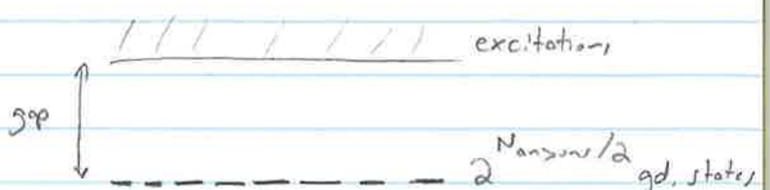
$\psi \to e^{i\alpha}\psi$ (Abelian anyons)

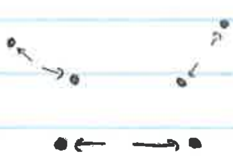
Focus here $\longrightarrow$ $\boxed{\psi_i \to U_{ij}\psi_j \text{ (non-Abelian anyons)}}$

(Ising) non-Abelian anyons

Three hallmarks:

1. Ground state degeneracy





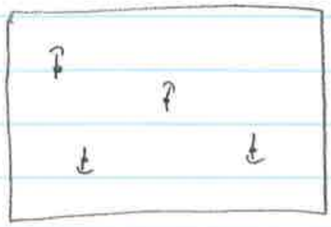
↑ locally indistinguishable, robust!
$\Rightarrow$ anyons carry unusual zero-energy degrees of freedom — "Majorana mode"

Fault-tolerant Qubits

[motivate through absence of dephasing]

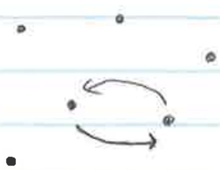
cf. free spin $\frac{1}{2}$'s:

insulator



$\Rightarrow 2^{N_{spin}}$ gd. states, but qualitatively different!
- locally distinguishable, not robust
[e.g., measure magnetizations, apply Zeeman field]

2. Non-Abelian statistics



adiabatic braid $\Rightarrow$ $\boxed{\psi_i \to U_{ij}\, \psi_j}$

braid matrix — rigid!

[i.e., depends on topology of braid, not detailed path chosen]

gd. states

Fault-tolerant Quantum Gates

3. Nontrivial fusion

can "fuse" in multiple ways

2 Ising anyons...

$\boxed{\sigma \times \sigma = I + \psi_F}$

... or form a fermion

... can annihilate

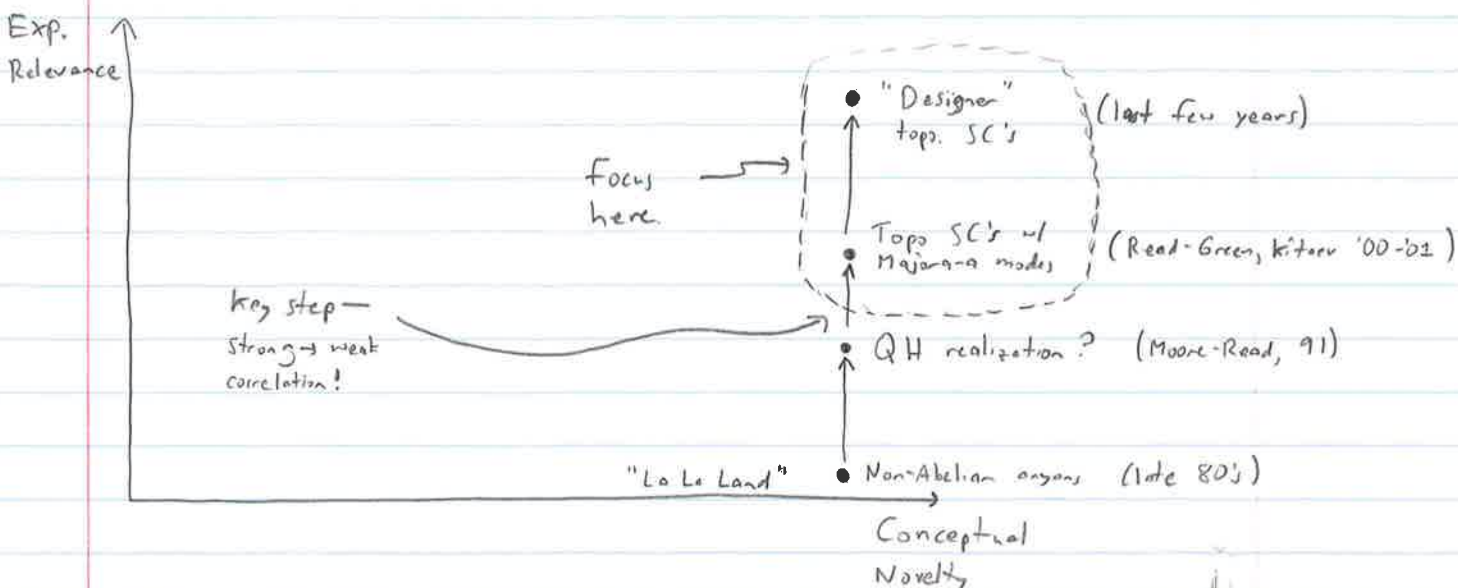
[Note that non-Abelian statistics $\Rightarrow$ nontrivial fusion + vice versa]

Readout

Kitaev — use for "topological quantum computation"!

But how to build hardware?

"Fisher plot"



Toy Models I: Kitaev chain

Spinless fermions on N-site chain w/ P.B.C.'s (for now).

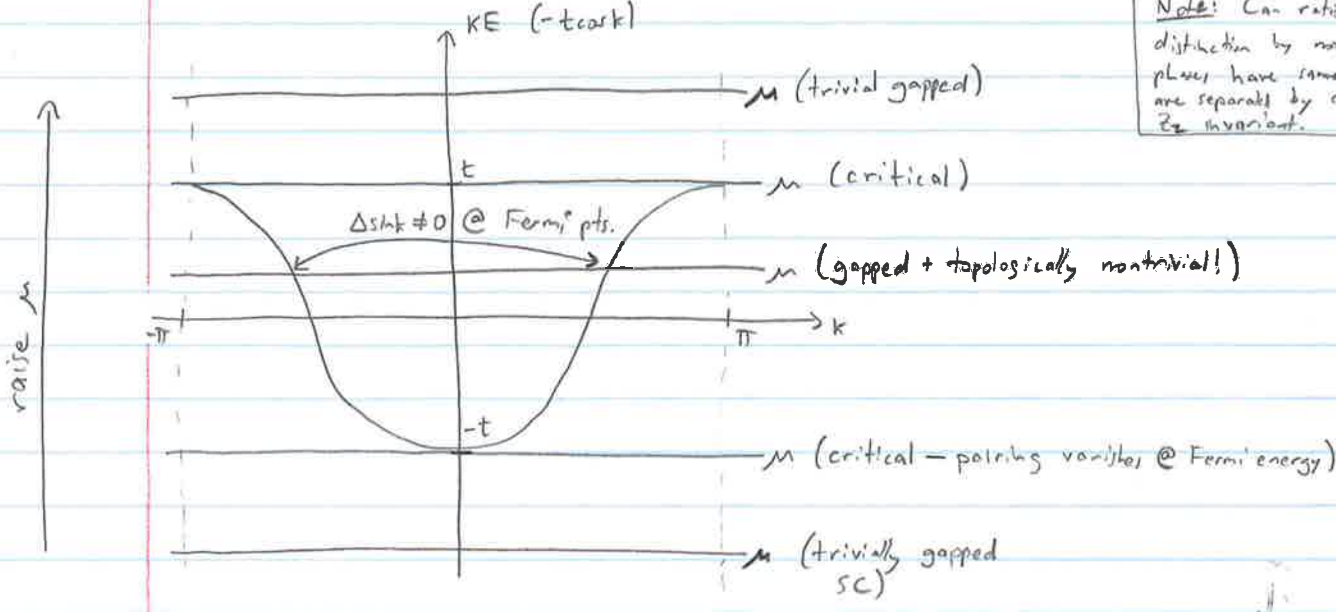
$$\boxed{H = \sum_x \left[-\mu c_x^\dagger c_x - \frac{1}{2}\left(t c_x^\dagger c_{x+1} + \Delta c_x c_{x+1} + h.c. \right) \right]}$$

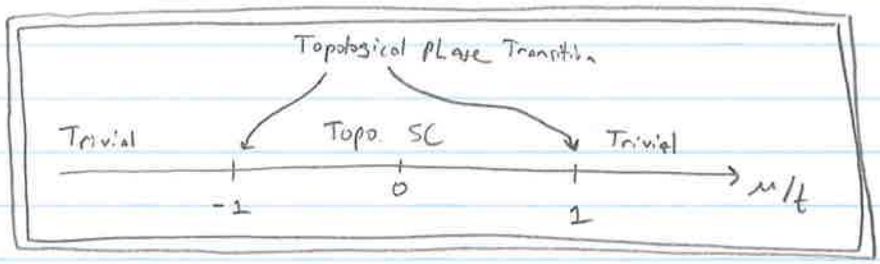
P.B.C. ⇒ go to k-space,

$$\boxed{H = \sum_{k \in BZ} \left[\left(-\mu - t\cos k \right) c_k^\dagger c_k + \left(\frac{i\Delta}{2} \underbrace{\sin k}_{\substack{\text{odd parity} \\ \text{(required by} \\ \text{spinlessness)}}} c_k c_{-k} + h.c. \right) \right]}$$

Phase Diagram as f$^{\underline{n}}$ of μ?

Note: Can rationalize topo distinction by noting that gapped phases have same symmetry, yet are separated by critical pt. Mention $\mathbb{Z}_2$ invariant.





Want to explore universal properties of phases / transitions. Convenient to take

$\boxed{\Delta = t}$ hereafter.

$$\Rightarrow \boxed{H = \sum_x \left[-\mu c_x^\dagger c_x - \tfrac{1}{2} t \left(c_x^\dagger + c_x \right)\left(c_{x+1} - c_{x+1}^\dagger \right) \right]}$$

hopping

pairing

Gapped phases

Take open B.C.'s now + use Majorana rep.:

$$\boxed{c_x = \frac{1}{2}\left(\gamma_{B,x} + i\gamma_{A,x}\right)}$$

Majorana ops.
(only pairs have well-defined occupation #'s)

$$\boxed{\gamma_\alpha = \gamma_\alpha^\dagger, \quad \gamma_\alpha^2 = I \qquad \{\gamma_\alpha, \gamma_{\alpha'\neq\alpha}\} = 0}$$

(Majorana op. algebra — reproduces $c_x^2 = (c_x^\dagger)^2 = 0$, $\{c_x, c_{x'}^\dagger\} = \delta_{x,x'}$)

[Note: this is always a legitimate rep. of any ordinary fermion op., like c_x. Does not however guarantee that a system supports Majorana-like excitations, as we'll see!]

Rewrite H:

$$c_x^\dagger c_x = \frac{1}{4}\left(\gamma_{B,x} - i\gamma_{A,x}\right)\left(\gamma_{B,x} + i\gamma_{A,x}\right)$$

$$= \frac{1}{4}\left(1 + 1 + i\gamma_{B,x}\gamma_{A,x} - i\gamma_{A,x}\gamma_{B,x}\right)$$

$$= \frac{1}{2}\left(1 + i\gamma_{B,x}\gamma_{A,x}\right)$$

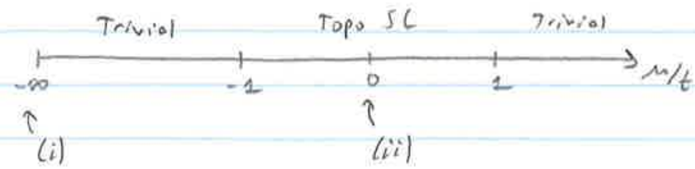
$$\boxed{(c_x^\dagger + c_x)(c_{x+1} - c_{x+1}^\dagger) = i\gamma_{B,x}\gamma_{A,x+1}}$$

$$\Rightarrow \boxed{H = \frac{-i}{2}\sum_x \left(\mu\gamma_{B,x}\gamma_{A,x} + t\gamma_{B,x}\gamma_{A,x+1}\right)}$$

c_x fermions

Majorana chain w/ competing dimerizations μ, t!

For revealing snapshots of gapped phases, examine 2 limits:

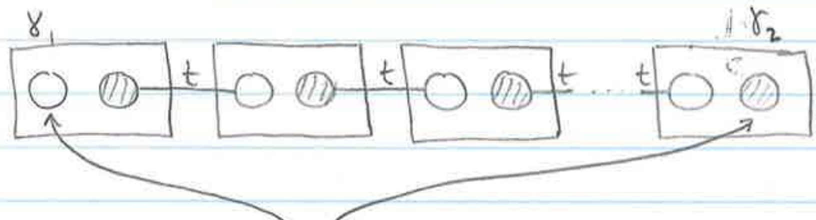


(i) $\mu < 0$, $t = 0$

Unique gd. state.

Trivial product state w/ no entanglement between sites. (Vacuum of c_x fermions.)

(ii) $\mu = 0$, $t > 0$



unpaired Majorana ops — zero modes!
[fermion split into two well-separated halves]

Several comments in order:

- $\boxed{\gamma_1 = \frac{c_1 - c_1^\dagger}{i}, \quad \gamma_2 = c_N + c_N^\dagger}$. $\boxed{[H, \gamma_{1,2}] = 0}$

- Non-local fermion $\boxed{d = \frac{\gamma_1 + i\gamma_2}{2}}$ can be filled, empty w/ no energy cost.

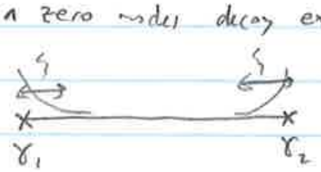
  $\Rightarrow$ 2-fold topological gd. state deg.!

  $\boxed{|0\rangle, \quad d^\dagger|0\rangle \equiv |1\rangle}$

  carry opposite fermion parity

  [Very unusual — most SC's prefer even parity so that all e^-'s can pair. Excitation energy for unpaired e^- vanishes for topo. reasons here.]

- For $\mu \neq 0$, $t \neq \Delta$ zero modes decay exponentially into bulk

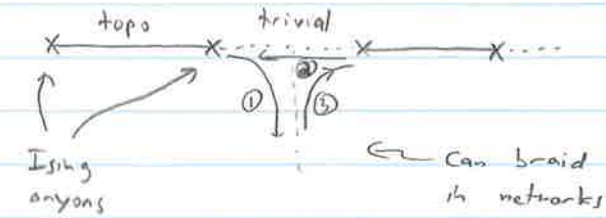


[fermion parity not locally detectable!]

- Deg. stable to local perturbations; no local measurement can distinguish gd. states. [contrast to deg. $\uparrow, \downarrow$ spin states]

- $\gamma_{1,2}$ are <u>not</u> particles (or quasiparticles)!

- Ends of topo SC $\simeq$ "Ising non-Abelian anyons". Zero-modes are "internal" deg of freedom that encode gd. state deg.

- 

Each anyon pair gives 2 gd. states

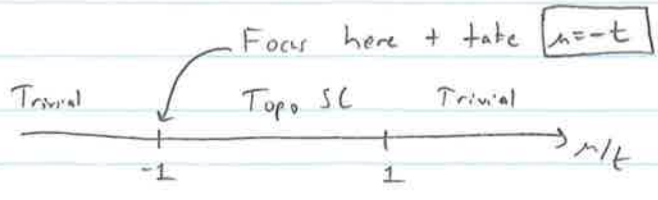
$$\boxed{\psi_i \to U_{ij} \psi_j}$$

[Braids effectively exchange "half" of one fermion w/ "half" of another — hence nontriviality.]

- 

Fusion channels!

Topo. Phase Transition



Low-energy physics @ criticality?

$$\Rightarrow \quad H \to \boxed{H_{crit} = \frac{-it}{2} \sum_x \left(-\gamma_{B,x} \gamma_{A,x} + \gamma_{B,x} \gamma_{A,x+1} \right) = \frac{-it}{2} \sum_x \gamma_{B,x} \left(\gamma_{A,x+1} - \gamma_{A,x} \right) \underset{\text{continuum limit}}{\sim} \frac{-it}{2} \int_x \gamma_B \partial_x \gamma_A}$$

Write $\boxed{\gamma_{A/B} = \gamma_R \pm \gamma_L}$ $\Rightarrow$ $H_{crit} = \frac{-it}{2}\int_x (\gamma_R - \gamma_L)\partial_x(\gamma_R + \gamma_L)$

$$= \frac{-it}{2}\int_x \Big(\gamma_R\partial_x\gamma_R - \gamma_L\partial_x\gamma_L + \underbrace{\gamma_R\partial_x\gamma_L - \gamma_L\partial_x\gamma_R}_{\text{cancel}}\Big)$$

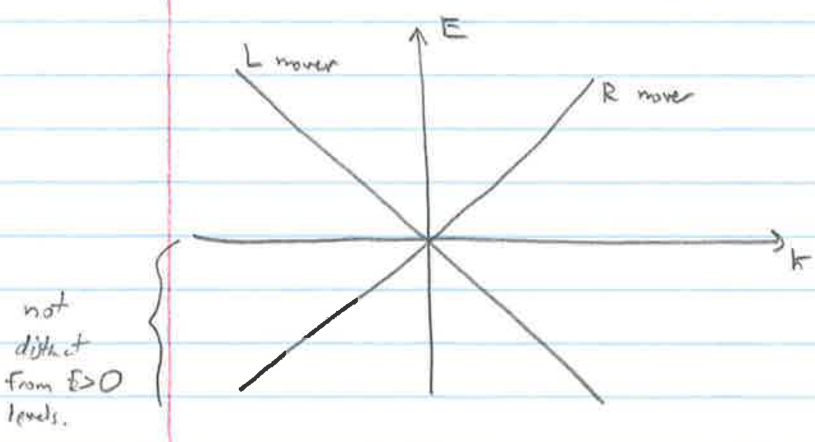
So we get $\boxed{\boxed{H_{crit} = \int_x \left(-iv\gamma_R\partial_x\gamma_R + iv\gamma_L\partial_x\gamma_L\right)}}$ $(v \propto t)$

chiral, gapless Majorana fermions!

Go to k-space: $\boxed{\gamma_{R/L}(x) = \int_k e^{ikx}\gamma_{R/L}(k)}$ [Implicitly thinking about P.B.C.'s again.]

Hermicity $\Rightarrow$ $\boxed{\gamma_{R/L}(k) = \gamma^\dagger_{R/L}(-k)}$ (*)

$$\Rightarrow \boxed{H_{crit} = \int_k \left[vk\,\gamma_R^\dagger(k)\gamma_R(k) - vk\,\gamma_L^\dagger(k)\gamma_L(k)\right]}$$



By (*), E>0, E<0 states not distinct!

$\Rightarrow$ "half" of usual single-channel wire.

[can be made very precise; e.g., thermal transport exactly half that in a usual wire ($c = \frac{1}{2}$ vs. 1).]

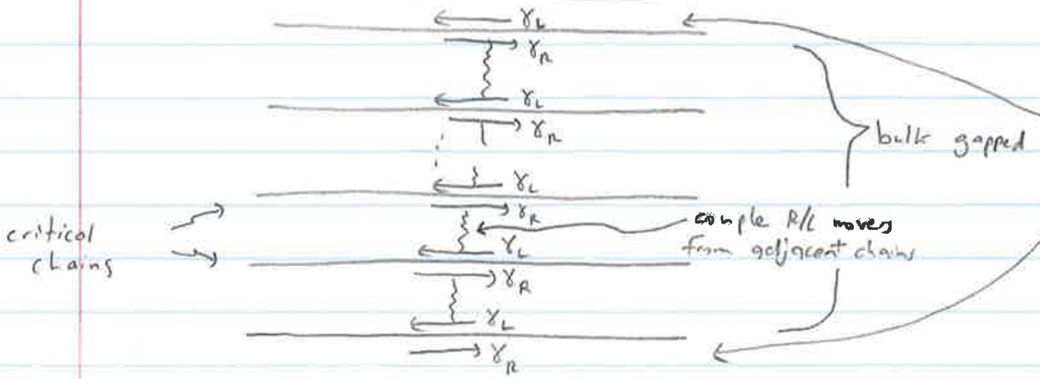
So Majorana physics in Kitaev chain appears in 2 ways:

(i) Localized zero modes in topo. phase

(ii) Gapless propagating deg. of freedom at criticality.

Toy Models II: 2D Topo. SC's

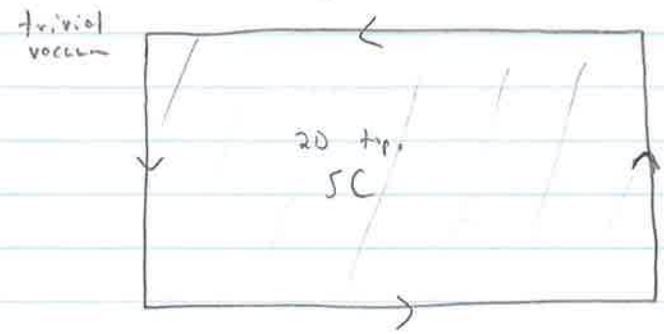
Build from array of <u>critical</u> Kitaev chains [efficiently leverages our understanding of 1D case.]



unpaired <u>chiral</u> Majorana edge states!

[hallmark of 2D topo SC. Note similarity to nontrivial dimerization in Kitaev chain.]

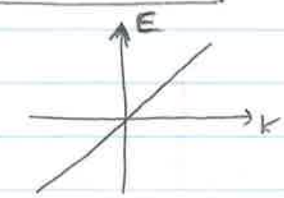
$\Downarrow$



edge coord. — chiral Majorana field

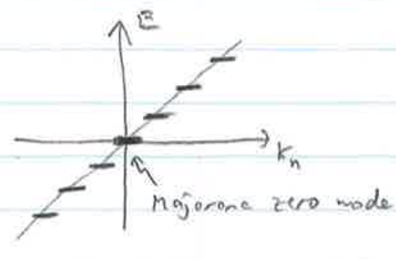
$$\boxed{H_{edge} = \int dx \left(-iv\,\gamma \partial_x \gamma\right)}$$

$$\boxed{E = vk}$$



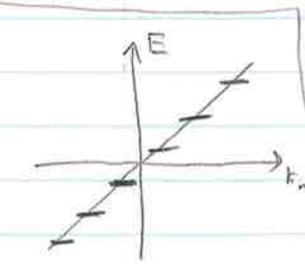
Important Q: Spectrum for finite perimeter L?

i.e. is k quantized to (i) $k_n = \frac{2\pi}{L} n$ (PBC)



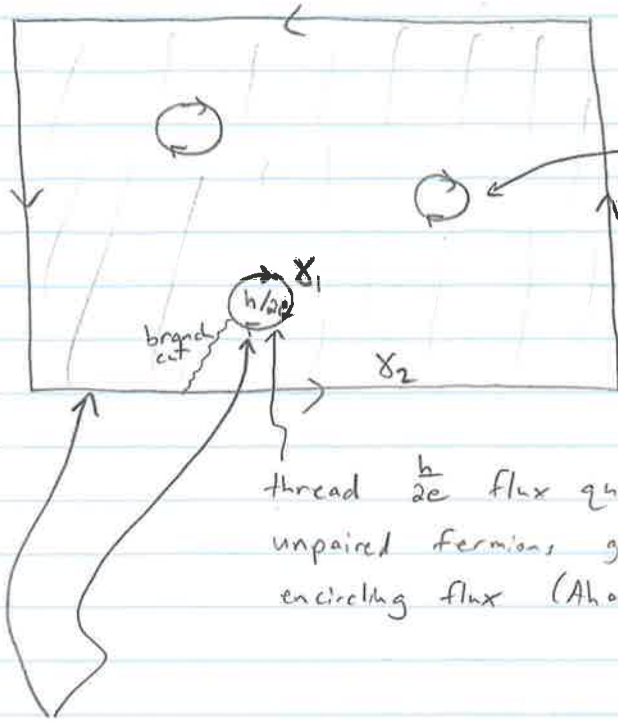
or

(ii) $k_n = \frac{2\pi}{L}(n + \frac{1}{2})$ (anti-PBC)



correct answer; PBC ruled out because you can't have just one Majorana zero mode! [Hilbert space wouldn't make sense.]

Drill holes:



any boundary binds chiral Majorana

thread $\frac{h}{2e}$ flux quantum; unpaired fermions get $-$ sign when encircling flux (Aharanov–Bohm)

Majorana B.C.'s shift from anti-periodic $\to$ periodic!

$\Rightarrow$ zero modes γ_1, γ_2!

Lessons: (i) boundaries of 2D topo SC host chiral Majorana fermions [cousins of Majorana end-states in 1D]

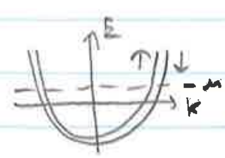
(ii) $\frac{h}{2e}$ flux localizes a Majorana zero mode [deeply related to (i)!] + $\therefore$ forms Ising non-Abelian anyon

<u>Experimental Blueprints</u>

Wanted: "Spinless" 1D, 2D SC's $\leftarrow$ [both harbor Ising anyons, albeit in different ways]

Challenges: (1) We live in 3D

(2) e^-'s carry spin



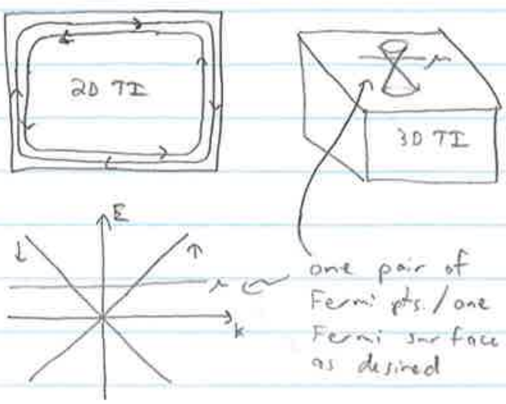
$\leftarrow$ Need to remove half of the carriers.

(3) Nearly all SC's arise from spin singlet Cooper pairs.

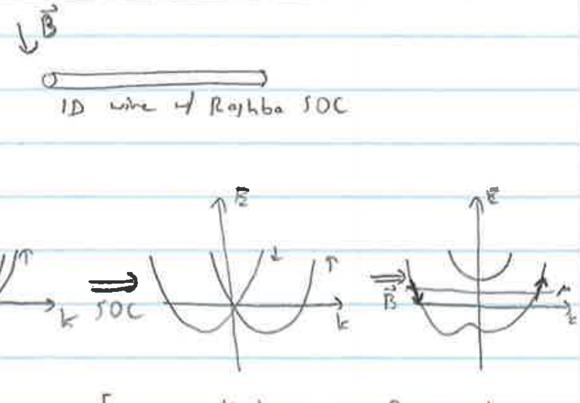
Likely no "intrinsic" realizations in solid state – despite 1000's of known SC's!

Can instead "engineer" topo SC's! O(100) papers revealing various strategies. Most follow a common recipe:

Step I Use TI boundary or Break $\mathcal{T}$ in low-D systems w/ SOC



e.g.,

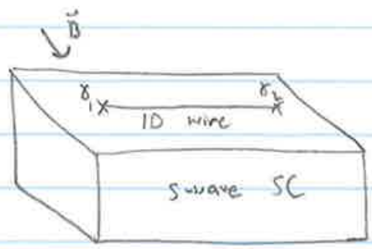


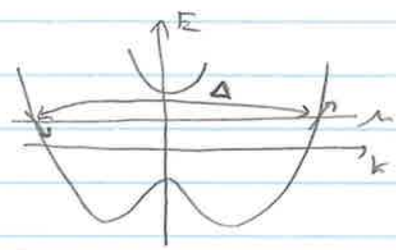
[crucial that carriers @ μ not spin polarized; c.f. 2D TI]

Solves challenges 1, 2! [in a way that makes challenge 3 "easy"]

Step II Couple systems above to s-wave SC.

e.g.,





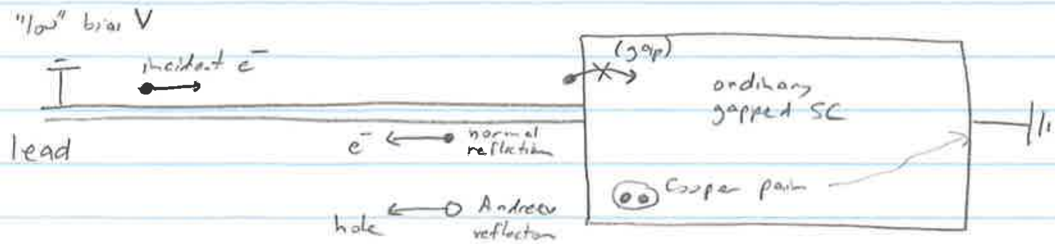
"proximity effect" drives 1D, 2D topo SC!!

Lots of expts followed, but first...

Majorana Detection

Focus on tunneling methods — most common so far.

Primer — SC tunneling as scattering problem

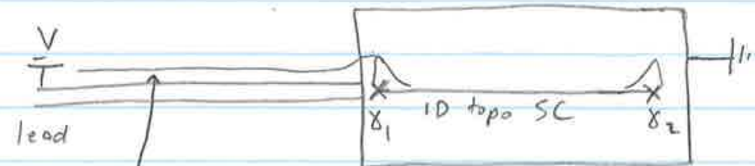


Conductance $\boxed{G = \frac{dI}{dV} = 2 \times \frac{e^2}{h} \times (\text{Andreev ref prob.})}$

reflects pair injection — conductance quantum

Solve just like potential barrier scattering in intro QM
— get wavefn, extract normal/Andreev ref. coefficients.

Result for topo SC



γ_1 delocalizes into lead

[generic property of localized modes coupled to gapless deg. of freedom]

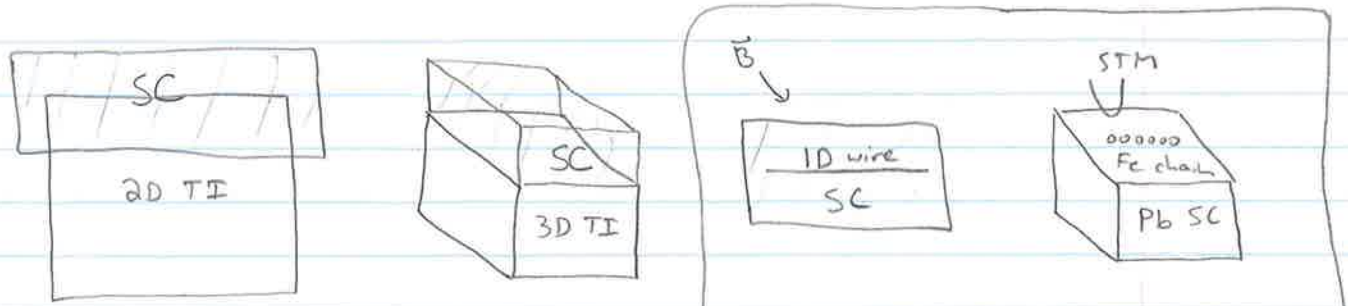
$\Rightarrow$ low-energy wavefn acquire equal e^-/hole character

$\Rightarrow$ Majorana-mediated "perfect Andreev reflection"!

$$\boxed{G = \frac{2e^2}{h} \quad (V \to 0)}$$

Experimental Status

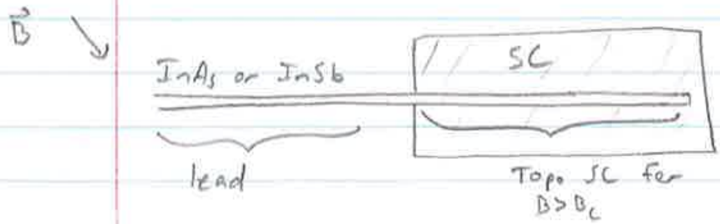
Devices built:

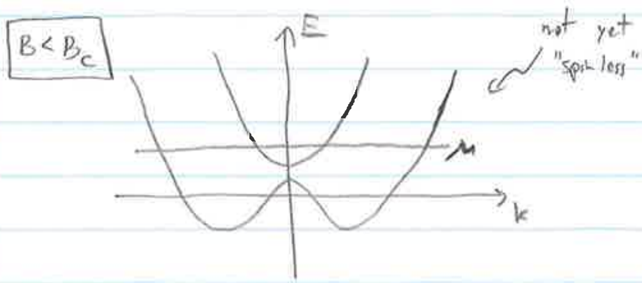


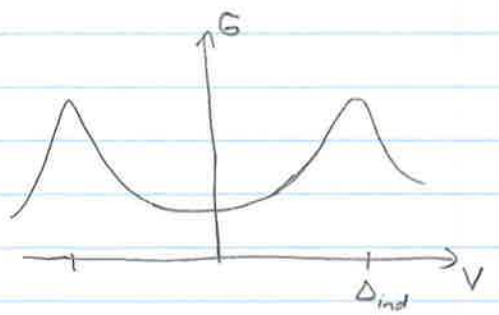
(Yacoby, Kouwenhoven, Du, ...) (van Harlingen, Hasan, ...) (Kouwenhoven, Heiblum, Marcus, ...) (Yazdani, Berlin group)

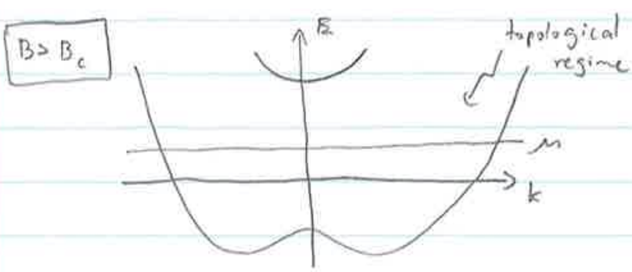
↑ Majorana evidence via tunneling here.

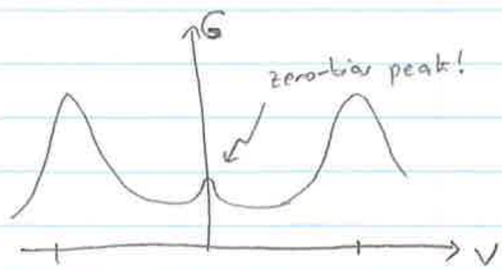
1D wire expts









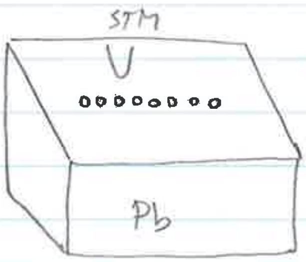


Consistent w/ Majorana zero-mode formation, but

- peak height far below $\frac{2e^2}{h}$ in most expts. [except recent unpublished data]
- signatures of phase transition often absent
- other explanations proposed [but harder sell as device quality improves]

My take: analogous features in latest high quality devices likely of Majorana origin. Reasonable alternatives not obvious.

Fe chain expts



Huge virtue: spatial resolution of tunneling conductance.

Zero-bias peaks localized to edges.

Tantalizing Majorana evidence! Future expt./theory will be exciting.

Future Milestones