QFT fun on the IBM QX
Building QFT Circuits from Hadamards

• 1-qubit Hadamard exchanges Z and X axes
  • Z and X are subject to an uncertainty relation like time and frequency → Fourier transform!

• $m$-qubit Hadamard is equivalent to Hadamards on each individual qubit
  • Can be recursively defined, with $H_0 = 1$ and

$$H_m = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix}$$

• $m$-qubit QFT is constructed from Hadamard gates and controlled phase gates

• Example: 2-qubit QFT in matrix and circuit form (verify they are equivalent!)

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

where $R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$
More on the QFT

• QFT is a key building block for many quantum algorithms
  • Period finding, phase estimation, factoring, ...

• Actual signal processing is **not** among these!
  • QFT performs the Fourier transform of a quantum state itself, not of a signal represented by qubits in $|0\rangle$ and $|1\rangle$ states

• Nonetheless, useful intuition may be gained from a toy example in which we encode various “signals” in the states of a set of qubits
Encoding a signal to be decoded by QFT

- Imagine a periodic function on a time interval from 0 to 1 seconds
  - At time $t$, function has phase $\varphi(t) = 2\pi ft$
- Given an $m$-qubit register $|q_m...q_0\rangle$, divide the interval into $2^m$ steps
- Encode function’s phase at time $t = 1/2^{(m-n)}$ s in the relative phase between $|0\rangle$ and $|1\rangle$ on qubit $q_n$: $|q_n\rangle = (|0\rangle + e^{2\pi ft}|1\rangle)/\sqrt{2}$
  - Phase represented by position on equator of Bloch sphere
- Preparing qubit register $|q_m...q_0\rangle$ in this way and then applying $m$-qubit Hadamard puts answer bit $i$ on qubit $m-i$ (read bits backwards)
Physical intuition for the QFT: 1 qubit

\( t = 1/2^{(1-n)}: |q_n\rangle = (|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2} \)

- \( f = 0 \) Hz:
  - \( t = 1/2: |q_0\rangle = (|0\rangle + e^{0i}|1\rangle)/\sqrt{2} \rightarrow H \)

- \( f = 1 \) Hz:
  - \( t = 1/2: |q_0\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2} \rightarrow H + Z \)
QX demo: $m$-qubit QFT, $m = 1$

For each frequency $f < 2^m$ Hz:
1. Prepare each qubit $q_n$ in $(|0⟩ + e^{2πift}|1⟩)/\sqrt{2}$, $t = 1/2^{(m-n)}$
2. Perform QFT
3. Measure all $q_n$
4. Reverse the bit order (in principle this could be done with a series of SWAP gates prior to measurement)
5. Convert binary to decimal; should recover $f$

<table>
<thead>
<tr>
<th>Freq.</th>
<th>Step:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>$0$ Hz</td>
<td>$q[0]$</td>
<td>$H$</td>
<td>$H$</td>
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<td></td>
<td>$0 \rightarrow 0 \rightarrow 0$</td>
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<tr>
<td>$1$ Hz</td>
<td>$q[0]$</td>
<td>$H$</td>
<td>$Z$</td>
<td>$H$</td>
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<td>$1 \rightarrow 1 \rightarrow 1$</td>
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</table>
Physical intuition for the QFT: 2 qubits

\[ t = \frac{1}{2^{(2-n)}}: \langle q_n \rangle = (\langle 0 \rangle + e^{2\pi i ft} \langle 1 \rangle) / \sqrt{2} \]

- **f = 0 Hz:**
  - \( t = \frac{1}{4}: \langle q_0 \rangle = (\langle 0 \rangle + e^{0i} \langle 1 \rangle) / \sqrt{2} \rightarrow H \)
  - \( t = \frac{1}{2}: \langle q_1 \rangle = (\langle 0 \rangle + e^{0i} \langle 1 \rangle) / \sqrt{2} \rightarrow H \)

- **f = 1 Hz:**
  - \( t = \frac{1}{4}: \langle q_0 \rangle = (\langle 0 \rangle + e^{i\pi/2} \langle 1 \rangle) / \sqrt{2} \rightarrow H + S \)
  - \( t = \frac{1}{2}: \langle q_1 \rangle = (\langle 0 \rangle + e^{i\pi} \langle 1 \rangle) / \sqrt{2} \rightarrow H + Z \)

- **f = 2 Hz:**
  - \( t = \frac{1}{4}: \langle q_0 \rangle = (\langle 0 \rangle + e^{i\pi} \langle 1 \rangle) / \sqrt{2} \rightarrow H + Z \)
  - \( t = \frac{1}{2}: \langle q_1 \rangle = (\langle 0 \rangle + e^{2i\pi} \langle 1 \rangle) / \sqrt{2} \rightarrow H \)

- **f = 3 Hz:** *(left as an exercise)*
QX demo: $m$-qubit QFT, $m = 2$

For each frequency $f < 2^m$ Hz:

1. Prepare each qubit $q_n$ in ($|0\rangle + e^{2\pi if t}|1\rangle)/\sqrt{2}$, $t = 1/2^{(m-n)}$
2. Perform QFT
3. Measure all $q_n$
4. Reverse the bit order (in principle this could be done with a series of SWAP gates prior to measurement)
5. Convert binary to decimal; should recover $f$

Freq. | Step: | 1 | 2 | 3 | 4 | 5
--- | --- | --- | --- | --- | --- | ---
0 Hz: | 00 → 00 → 0
1 Hz: | 10 → 01 → 1
2 Hz: | 01 → 10 → 2
3 Hz: (left as an exercise)
Physical intuition for the QFT: 3 qubits

\[ t = \frac{1}{2^{3-n}}: \quad |q_n\rangle = (|0\rangle + e^{2\pi i f t} |1\rangle)/\sqrt{2} \]

- \( f = 0 \) Hz: (still trivial: \( H \) on all 3 qubits)
- \( f = 1 \) Hz:
  - \( t = 1/8: \quad |q_0\rangle = (|0\rangle + e^{i \pi /4} |1\rangle)/\sqrt{2} \rightarrow H + T \)
  - \( t = 1/4: \quad |q_1\rangle = (|0\rangle + e^{i \pi /2} |1\rangle)/\sqrt{2} \rightarrow H + S \)
  - \( t = 1/2: \quad |q_2\rangle = (|0\rangle + e^{i\pi} |1\rangle)/\sqrt{2} \rightarrow H + Z \)
- \( f = 2 \) Hz:
  - \( t = 1/8: \quad |q_0\rangle = (|0\rangle + e^{i \pi /2} |1\rangle)/\sqrt{2} \rightarrow H + S \)
  - \( t = 1/4: \quad |q_1\rangle = (|0\rangle + e^{i\pi} |1\rangle)/\sqrt{2} \rightarrow H + Z \)
  - \( t = 1/2: \quad |q_2\rangle = (|0\rangle + e^{2i\pi} |1\rangle)/\sqrt{2} \rightarrow H \)
- \( f = 3 \) Hz:
  - \( t = 1/8: \quad |q_0\rangle = (|0\rangle + e^{3i \pi /4} |1\rangle)/\sqrt{2} \rightarrow H + S + T \)
  - \( t = 1/4: \quad |q_1\rangle = (|0\rangle + e^{3i \pi /2} |1\rangle)/\sqrt{2} \rightarrow H + S^* \)
  - \( t = 1/2: \quad |q_2\rangle = (|0\rangle + e^{i\pi} |1\rangle)/\sqrt{2} \rightarrow H + Z \)
- \( f > 3 \) Hz: (left as an exercise)
QX demo: $m$-qubit QFT, $m = 3$

For each frequency $f < 2^m$ Hz:
1. Prepare each qubit $q_n$ in
   \[ |0\rangle + e^{2\pi ift} |1\rangle \] / $\sqrt{2}$, $t = 1/2^{(m-n)}$
2. Perform QFT
3. Measure all $q_n$
4. Reverse the bit order (in principle this could be done with a series of SWAP gates prior to measurement)
5. Convert binary to decimal; should recover $f$

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<tr>
<td>0 Hz</td>
<td>000</td>
<td>000</td>
<td>0</td>
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</tr>
<tr>
<td>1 Hz</td>
<td>100</td>
<td>001</td>
<td>1</td>
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<td>010</td>
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<tr>
<td>3 Hz</td>
<td>110</td>
<td>011</td>
<td>3</td>
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> 3 Hz: (left as an exercise)
QFT Summary

• Single qubit: Hadamard equivalent to quantum Fourier transform
• Many qubits: QFT composed of Hadamards and controlled-phase gates
• QFT transforms quantum states, not signals
• Enables a measurement in the computational basis to reveal information about phase relationships
• Next step: see how the QFT figures into important quantum algorithms like period finding and phase estimation